PROPAGATION OF A LIGHT RAY THROUGH A CONTINUOUS ABERRATIONAL LENTICULAR MEDIUM

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An equation is derived for the trajectory of a light ray through a lenticular medium where the refractive index decreases with increasing distance from some line, but not according to the square law, and an approximate solution to this equation is found for the cases of constant curvature and oscillating curvature of that line.

The propagation of electromagnetic waves within the optical range of the spectrum through an electrically neutral medium with a slowly varying refractive index is described, without accounting for diffractive and quantum effects, by the well-known equation of an eikonal:

$$\frac{d}{ds}(ns) = \nabla n.$$

We will consider a medium whose refractive index n is constant and equal to n_0 along some given line l in space but varies with the distance ρ from this line:

$$n = n_0 + \varphi(\rho); \qquad \varphi(\rho)|_l = 0.$$

The medium may be any continuous one and the function $\varphi(\rho)$ is generated by some technical process such as heating, for example.

The following equation has been derived in [1]

$$\frac{d^2\mathbf{r}}{dz^2} = \left.\frac{d\ln n}{d\rho}\right|_{\rho=r}\frac{\mathbf{r}}{r} + \frac{\mathbf{R}_c}{R_c^2} \tag{1}$$

to describe the propagation of the center ray in a light beam through media with such a refractive index.

Introducing here the system of local coordinates

$$\mathbf{r} = x\frac{\mathbf{R}_c}{R_c} + y\mathbf{e}_y$$

(with the unit vector $\mathbf{e}_y$ and the orthogonal vector $\mathbf{R}_c$), we obtain the following system of equations:

$$\frac{d^2x}{dz^2} = \left.\frac{d\ln n}{d\rho}\right|_{\rho=r}\frac{x}{r} + \frac{1}{R_c};$$
$$\frac{d^2y}{dz^2} = \left.\frac{d\ln n}{d\rho}\right|_{\rho=r}\frac{y}{r}. \tag{2}$$

These equations are valid for any function $\varphi = \varphi(\rho)$. If φ is a quadratic function of ρ, then the medium is called aberrationless. The case of an aberrationless medium has been analyzed thoroughly in [1]. The equations for the trajectory of a ray are linear then. If φ is a nonquadratic function of ρ, however, then the

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medium is called aberrational and the equations become nonlinear, which complicates the problem. In this study the authors will analyze the aberrational case, namely a medium with so-called spherical aberration. We stipulate $\varphi(\rho)$ as follows:

$$\varphi(\rho) = -\frac{\alpha}{2}\rho^2 + \frac{\beta}{4}\rho^4; \quad \alpha > 0 \tag{3}$$

assuming the specific convergence of the structure α and the so-called aberration parameter β to be of such magnitudes that

$$\alpha r_0^2 \approx \beta r_0^4 \ll 1,$$

where r_0 denotes some effective radius in the medium. We will also assume that the curvature of line l is only small, namely

$$\frac{r_0}{R_c} \ll 1.$$

On this premise, system (2) becomes

$$\begin{aligned} \frac{d^2x}{dz^2} &= -\alpha x + \beta(x^2+y^2)x + \frac{1}{R_c(z)}; \\ \frac{d^2y}{dz^2} &= -\alpha y + \beta(x^2+y^2)y. \end{aligned} \tag{4}$$

We will now consider only the projection of the ray trajectory on the inflection plane $y = 0$ of line l, so that Eqs. (4) reduce to the equation

$$\frac{d^2x}{dz^2} = -\alpha x + \beta x^3 + \frac{1}{R_c(z)}. \tag{5}$$

With the curvature of line l given as $F(z) = 1/R_c(z)$ and with the initial conditions

$$\begin{aligned} x\big|_{z=0} &= x_0; \\ \frac{dx}{dz}\bigg|_{z=0} &= \dot{x}_0, \end{aligned} \tag{6}$$

we can now solve Eq. (5).

Let us analyze Eq. (5) for two special cases of function F(z). Since this equation is nonlinear, but its nonlinearity has been assumed "weak," hence it may be analyzed by the methods of nonlinear mechanics.

1. Constant Curvature, $F(z) = 1/R_c^0$. The equation becomes

$$\frac{d^2x}{dz^2} = -\alpha x + \beta x^3 + \frac{1}{R_c^0}. \tag{7}$$

It can be analyzed either by the Moiseev method [2] or by changing the variable according to Hayasi [3], and then be reduced to the well-known Duffing equation. Inasmuch as both approaches yield the same result, to the first approximation, we will take the second simpler one. We change to a new variable $u = x + 1/\alpha R_c^0$ and disregard all terms $\beta/(\alpha R_c^0)^2$, $\beta/(\alpha R_c^0)^3$ of higher-order smallness. Now Eq. (7) becomes

$$\frac{d^2u}{dz^2} + \alpha u = \beta\left(u^3 + \frac{3u^2}{\alpha R_c^0}\right),$$

and the solution to this one, to the first approximation, can be written as in [4]

$$u = w\cos\left[\left(\sqrt{\alpha} - \frac{3\beta w^2}{8\sqrt{\alpha}}\right)z + C\right], \tag{8}$$

with a constant amplitude ω and with the frequency depending on the amplitude so as not to result in isochronous oscillations. Relation (8) indicates that oscillations of the light ray will be sinusoidal with a decreasing period when the parameter β is negative and with an increasing period when it is positive, $\beta = 0$

representing the aberrationless case. With the initial conditions as stipulated in (6), it is not difficult to find the maximum deflection of the light ray from the curve $F(z) = 1/R_c^0$.

Equation (7) belongs to the class of differential equations whose solutions are constants, as it is well known [5] that for equations of the kind

$$\Phi(x^{(n)}, x^{(n-1)}, \ldots, x', x, z) + G(x) + B = 0, \tag{9}$$

with $\Phi|_{x=\text{const}} = 0$, $G(x) \neq 0$, and $B \neq 0$, the initial conditions

$$x|_{z=0} = x_i; \quad x'|_{z=0} = \ldots = x^{(n-1)}|_{z=0} = 0 \tag{10}$$

represent solutions to Eq. (9) if x_i are the roots of the equation

$$G(x) + B = 0.$$

This simple fact in the theory of differential equations leads to a very interesting consequence here: let us impose on Eq. (7) the initial conditions

$$x|_{z=0} = x_i; \quad \left.\frac{dx}{dz}\right|_{z=0} = 0;$$

and let x_i be the roots of the equation

$$x^3 - \frac{\alpha}{\beta} x + \frac{1}{\beta R_c^0} = 0.$$

Since this is the third-degree equation in x, hence at least one root must be real. The existence of such a root indicates that a ray from point x_1 tangent to the curve $F(z) = 1/R_c^0$ will proceed congruently to this curve. With a small displacement δ from x_1, the ray will obviously oscillate about the curve $R_c^0 + x_1$ at an amplitude approximately equal to δ. For an aberrationless medium this constant solution is obviously equal to $1/\alpha R_c^0$.

2. Oscillating Curvature. For this case we stipulate function F(z) as

$$F(z) = \frac{1}{R_c^0} \sin \nu z$$

and Eq. (5) takes the form

$$\frac{d^2x}{dz^2} = -\alpha x + \beta x^3 + \frac{1}{R_c^0} \sin \nu z. \tag{11}$$

For the assumptions made with respect to α, β, and R_c, Eq. (11) is the standard equation of nonlinear mechanics with small parameters and an "external periodic force." It is known [4], that in the presence of an external periodic force there is a resonance and a nonresonance case. In the nonresonance case the effect of the external force appears only in the second approximation; in the first approximation the perturbing force can be neglected. In the resonance case there is only the principal resonance $\alpha = \nu^2$. Since it is this case which makes the main contribution to the disturbance of the beam, we consider Eq. (11) for $\nu = \sqrt{\alpha}$. The solution to (11) in the first approximation can be written in the form [4, 6]

$$x = A \sin \nu z + \frac{\beta A^3}{32\nu^2} (\sin \nu z - \sin 3\nu z), \tag{12}$$

where the amplitude A is determined by the relation

$$\frac{3}{4} \beta A^3 - \frac{1}{R_c^0} = 0. \tag{13}$$

Solution (12) is bounded, unlike the solution to Eq. (11) for the ray trajectory in an aberrationless medium ($\beta = 0$) at resonance

$$x = \frac{1}{2\alpha R_c^0} [\sqrt{\alpha}\, z \cos \sqrt{\alpha}\, z - \sin \sqrt{\alpha}\, z],$$

the latter yielding an unbounded ray deflection from the curve $F(z) = \sin \nu z/R_c^0$.

Thus, the stability of light rays propagated through certain aberrational lenticular media is higher than in media with a square-law relation for the refractive index. Accordingly, it is sometimes preferable to produce synthetic lens materials with nonquadratic characteristics.

NOTATION

n is the refractive index;
s is the arc length;
$\mathbf{s}$ is the unit vector tangent to the ray trajectory;
l is a line in space;
ρ is the distance from line l;
$\mathbf{r}$ is the radius-vector of a point on the ray;
z is the curvilinear longitudinal coordinate on line l;
$\mathbf{R}_c$ is the radius of curvature of line l;
x is the radial coordinate in the direction of $\mathbf{R}_c$;
y is the coordinate in the direction perpendicular to $\mathbf{R}_c$ and $\mathbf{s}$;
α is the specific convergence of the structure;
β is the aberration parameter.

LITERATURE CITED

1. H.-G. Unger, Archiv Elektrischer Übertragung, 19, 189 (1965).
2. N. N. Moiseev, Asymptotic Methods of Nonlinear Mechanics [in Russian], Nauka (1969).
3. T. Hayasi, Nonlinear Oscillations in Physical Systems [Russian translation], Mir (1968).
4. N. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in the Theory of Nonlinear Oscillations [in Russian], Izd. FM (1963).
5. V. V. Stepanov, Study Course in Differential Equations [in Russian], GITTL (1953).
6. J. Stocker, Nonlinear Oscillations in Mechanical and Electrical Systems [Russian translation], IL (1953).